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APPLICATION AND NUMERICAL SOLUTION OF ABEL-TYPE INTEGRAL EQUATIONS

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APPLICATION AND NUMERICAL SOLUTION OF ABEL-TYPE INTEGRAL EQUATIONS*

R. S. Anderssen†

Dedicated to Professor Arvid T. Lonseth on his 65th birthday.

Technical Summary Report # 1787
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ABSTRACT

A commonly occurring class of integral equations which arise regularly in applications are the separable first kind Abel-type integral equations such as

$$(*) \qquad s(y) = \int_y^\infty k_1(y)k_2(x)\,(x^2 - y^2)^{-\mu}u(x)dx, \quad 0 < \mu < 1, \quad x \geq y \geq 0 .$$

Though the mathematical properties of this equation (such as conditions for the existence, uniqueness and smoothness of its solutions, its improperly posed nature, the existence of inversion formulas, etc.) have been examined in some detail in the literature, its numerical solution poses a number of difficulties especially when $s(y)$ is only available as discrete observational data

$$\{d_i\} = \{d_i = s(x_i) + \varepsilon_i,\ i = 1,2,\ldots,n;\ \varepsilon_i \text{ discrete random errors}\} .$$

In this paper, we (i) review the numerous applications in which the solutions of equations like (*) for discrete observational data $\{d_i\}$ is the basic step, (ii) with respect to given discrete observational data $\{d_i\}$, compare the use of pseudo-analytic methods and the direct evaluation of its inversion formulas as a basis for solving (*), (iii) propose a specific algorithm based on the conclusions of (ii), and (iv) examine the consequences of the fact that, for (*),

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linear functionals defined on its solution $u(x)$ can be redefined as linear functionals on the data $s(y)$. The justification for the latter is that, in applications involving separable first kind Abel-type integral equations, inferences are usually based on (linear) functionals defined on $u(x)$, not on $u(x)$ itself. This point is illustrated with an example from metallurgy.

AMS (MOS) Subject Classifications: 45E10, 45L10, 60D05, 60J65, 62P99, 65D25, 65D30, 65R05, 86A15

Key Words: Abel-type integral equations, Brownian motion, geometric probability, improperly posed, interferometry, inversion formulas, linear functionals, metallurgy, microscopy, particle size distribution, product integration, pseudo-analytic methods, random spheres model, regularization, seismology, spectral differentiation (maximum likelihood), stereology, Volterra integral equation, Wiener filtering, X-ray scattering

Work Unit Number 2 (Other Mathematical Methods)

SIGNIFICANCE AND EXPLANATION

In general, it is not possible to determine three-dimensional structure (such as the distribution of carbon particles in steel, the size, shape and position of weld defects in welded thick steel plates, and the distribution of cells in and structure of biological and zoological specimens) directly. It is necessary to make indirect observations such as are obtained by taking X-ray photographs (two-dimensional projections) of the three-dimensional structure, or by cutting open (taking (random) two-dimensional (planar) sections of) the three-dimensional structure and photographing, or by making microscope slides ((small) thin sections of the three-dimensional structure) for examination in a (electron) microscope. Thus, it is necessary to determine information (properties) about the original three-dimensional structure from the two-dimensional information (data) obtained from the indirect observations (such as the X-ray photographs, photographs of planar sections, or (electron) microscope photographs). In a number of situations (for example, when the three-dimensional structure involves axial or spherical symmetry), the mathematical formalism which relates properties of the three-dimensional structure $u(x)$ to corresponding two-dimensional data $s(y)$ is the following separable first kind Abel-type integral equation

$$(*) \qquad s(y) = \int_y^\infty k_1(y)k_2(x)\,(x^2 - y^2)^{-\mu}u(x)dx, \quad 0 < \mu < 1, \quad x \geq y \geq 0 ,$$

where $k_1(y)$, $k_2(x)$ and μ are known. For example, if the three-dimensional structure consists of randomly dispersed (approximately) spherical carbon particles in steel, and $u(x)$ is the size (radius or diameter) distribution of these sph[illegible]al particles, then, with $k_1(y) = y/m$, $m = \int_0^\infty xu(x)dx$, $k_2(x) = 1$, and $\mu =$ [illegible]) corresponds to the random spheres model of stereology with $s(y)$ the size distribution of the circular sections of the spherical carbon

particles on random plane sections. In fact, the mathematical formalisms for a large range of applications are defined by (*). A comprehensive classification for them is given in the paper.

Thus, in the above-mentioned situations, the determination of the required three-dimensional property $u(x)$ reduces to solving (*) for some given $s(y)$. But, in application, $s(y)$ is not usually known exactly, but is only available as discrete observational data

$$d_i = s(x_i) + \varepsilon_i, \quad i = 1,2,\dots,n ,$$

where the ε_i denote discrete random errors. The paper therefore examine in some detail the question of how to solve (*) in such situations. In addition, because of its practical numerical consequences, the evaluation of linear properties (functionals) (such as mean surface area, mean volume, etc.), defined on the solution $u(x)$, of the form

$$\int_y^\infty \theta(x)u(x)dx , \quad 0 \le y < \infty ,$$

where $\theta(x)$ is known, is discussed in some detail with respect to a specific example from metallurgy (namely, the determination of the impact strength of a given consignment of steel).

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PREFACE

This paper is the basis for the invited talk given at the Seminar organized by the North-West Division of SIAM to honour the retirement of Professor Arvid T. Lonseth, Mathematics Department, Oregon State University, Corvallis. It was held in Seattle on August 15 and 16 in parallel with the 81st Summer Meeting of the American Mathematics Society.

Support from the Computer Centre, ANU (Australian National University), and the Mathematics Research Center, University of Wisconsin-Madison, which ensured that participation would be possible, is gratefully acknowledged.

In compiling a review of Abel-type integral equations in applications, it has been necessary to discuss various points with colleagues from different disciplines to whom the author is indebted. They include Drs. Frank de Hoog (Computer Centre), John Cleary (Earth Sciences), Daryl Daley (Statistics), Pam Davey (Statistics), Roger Miles (Statistics) all of the ANU and Professor J. E. Hilliard (Material Sciences) at Northwestern University. The author wishes to record particular thanks to Tony Jakeman, CRES, ANU. The writing of this report has benefited substantially from lengthy discussions with him about numerous finer points and detail.

§1. Introduction: What is an Abel-type Integral Equation?

Integral equations of the form

$$s(y) = \int_y^Y K(y,x)\, u(x)dx, \quad 0 \le y \le x \le Y \le \infty, \tag{1.1}$$

(and its equivalent

$$s(\bar{y}) = \int_0^{\bar{y}} K(\bar{y},\bar{x})\, u(\bar{x})d\bar{x}, \quad 0 \le \bar{x} \le \bar{y} \le Y \le \infty,$$

with $\bar{y} = Y - y$, $\bar{x} = Y - x$) are known as *first kind Volterra integral equations* with *kernel* $K(y,x)$. The kernel $K(y,x)$ is said to be non-singular or singular depending on whether $K(y,x)$ is continuous or discontinuous on the triangular region $0 \le y \le x \le Y$. In the special situation where $K(y,x)$ takes the form

$$K(y,x) = k(y,x) \,/\, (x^p - y^p)^\mu, \quad 0 \le \mu < 1, \ p > 0, \tag{1.2}$$

with $k(y,x)$ continuous on $0 \le y \le x \le Y$ and $k(y,y) \ne 0$, the equations (1.1) are called *generalized first kind Abel integral equations* or *first kind Abel-type integral equations*.

The reason for the name is easily explained. When he posed and solved in 1823 the falling-particle problem (see Lonseth(1977)) as a generalization of Huygen's problem of the isochronous pendulum (1673), Abel obtained the equation

$$\int_0^y u(x)\, (y - x)^{-\frac{1}{2}}\, dx = \sqrt{2g}\; T(y) \tag{1.3}$$

which now bears his name, where $T(y)$ denotes the time for a particle to fall without friction a distance y, g the Earth's gravitational constant, and $u(x)$ the path the particle must be constrained to fall in, if its time of fall is to coincide with some prescribed function $T = T(y)$ of the distance fallen. The structural similarity between (1.1), with $K(y,x)$ defined by (1.2), and (1.3) is obvious. In fact, the similarity goes deeper than this (see, for example, Atkinson (1974)) and is the main reason for the definitions. The pertinence of this point will become clearer in the sequel.

As we shall see below, a very important subclass of first kind Abel-type integral equations is generated when $k(y,x)$ is *separable*, namely

$$k(y,x) = k_1(y)\, k_2(x). \tag{1.4}$$

For example, when (1.1) takes the form

$$s(y) = \int_y^\infty k_1(y)k_2(x)\, (x - y)^{-\mu}\, u(x)dx, \quad 0 < \mu < 1, \ x \ge y \ge 0, \tag{1.5}$$

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$u(x)$ is defined by the following inversion formulas, when $u(x)\, k_2(x)$ is continuous, $k_2(x) \neq 0$, $x \in [0,\infty)$, and $s(y)/k_1(y)$ is differentiable,

$$u(x) = -\,[\pi k_2(x)]^{-1} \sin \mu\pi \left[\frac{s(y)}{k_1(y)(y-x)^{1-\mu}} \right|_{y \to \infty} + \int_x^\infty (y-x)^{\mu-1} \frac{d}{dy}\left\{ \frac{s(y)}{k_1(y)} \right\} dy \Bigg], \tag{1.6a}$$

$$= -\,[\pi k_2(x)]^{-1} \sin(\mu\pi) \frac{d}{dx}\left\{ \int_x^\infty (y-x)^{\mu-1} \frac{s(y)}{k_1(y)}\, dy \right\}. \tag{1.6b}$$

We shall refer to this important subclass as *separable first kind Abel-type integral equations.*

Since the addition of the term $\lambda u(y)$, $\lambda \neq 0$, to the right hand side of (1.1) converts it to a *second kind Volterra equation,* namely

$$s(y) = \lambda u(y) + \int_y^Y K(y,x)u(x)dx, \quad 0 \leq y \leq x \leq Y \leq \infty, \tag{1.7}$$

we shall refer to the corresponding equations as *second kind Abel-type integral equations,* when the kernel $K(y,x)$ in (1.7) takes the form (1.2), and *separable second kind Abel-type integral equations* when the kernel $K(y,x)$ in (1.7) takes the form (1.2) and $k(y,x)$ is separable.

Though the Abel-type integral equations which arise most frequently in applications are of the first kind, the second kind are also found. In particular, they arise in the analysis of thin sections in transmission electron microscopy (e.g. Goldsmith (1967)). The most commonly occurring Abel-type integral equations in applications are certainly the separable first kind equations.

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§2. *Preliminaries and Notation*

Because it is impossible to cover in this paper all the major aspects associated with the theory and application of Abel-type equations, we cite briefly source information for material not explicitly discussed in, but closely related to the context of, the paper, as well as define notation and concepts required subsequently.

2.1 Existence and Uniqueness of Solutions of Abel-type Integral Equations.

Atkinson (1974) has shown that an existence theory can be constructed for Abel-type equations (in the form equivalent to (1.1)) when the kernel $K(y,x)$ has the very general form

$$K(y,x) = k(y,x) \ / \ (y^p - x^p), \ 0 \leq x < 1, \ p > 0,$$

with $k(y,x)$ continuous on $0 \leq x \leq y \leq Y$ and $k(y,y) \neq 0$. In addition, the paper is a source reference to previous work on the subject.

2.2 The Probabilistic Basis of Stereology

Excellent discussions of the geometric probability basis of the theory of estimating the properties of the distribution of one material in another by investigating the observed patterns on plane and line intersections (stereology) are now available:

(i) Moran (1972) for a lucid introduction.

(ii) Miles and Davy (1977a), (1977b), Davy and Miles (1977) and Davy (1977) for a rigorous examination of the probabilistic assumptions and structure.

(iii) Santalo (1953), (1955) for a lucid introduction to the mathematical foundations of geometric probability (namely, integral geometry) and for one of the key examinations for convex particles.

2.3 Preliminaries and Notation for the Estimation of Size Distributions

For some given three-dimensional object (such as a sample of steel, cancer infected tissue, a conglomerate of lipid cells, etc.), the aim is to derive information about the object's three-dimensional structure from lower dimensional observations of it (such as is obtained with linear probes as well as with planar and thin sections). For example, if the phase of interest in the object is particulate (such as lipid cells in a conglomerate, cancer cells in tissue or inclusions in steel), deductions about structure for application purposes may involve estimation of some or all of the following properties of that particulate phase: shape, number, volume, size, surface area and length. When simple logic is unable to yield the required information for subsequent inferences, mathematical methods based on geometric probability, statistics and numerical analysis are required to formulate and solve the equations relevant to assumptions made about the global geometry of the object.

In the sequel, because our interest is in Abel-type integral equations, we restrict attention to objects consisting of two-phases, α and β, with the α-phase made up of similarly-shaped convex particles and the β-phase the matrix in which the α-phase is embedded. It is assumed that the α-phase particles are

(i) convex,

(ii) similarly shaped (i.e. *definable in terms of only one parameter* x), *and*

(iii) randomly distributed in a convex field (i.e.; the shape of the object itself is also convex).

We denote these particles by $\{P_x\}$, with parameter x, and assume that their probability size density is $g(x) = G'(x)$ $(G(-\infty) = 0)$.

In addition, let the probability density of the observations be (a) = $Z'(a)$ where a denotes either area or length, depending on particle shape and the type of observation process implemented. Write a_n as the maximum over the sizes a. Also dependent on these two factors, use of the notation V or L will be required: let V be the volume of the particle P_1 (that is, the convex particle with size parameter $x = 1$), S its surface area and C its integral of mean curvature. When these quantities are not appropriate, we use L to denote length of P_1. Denote by N_L the average number of particles per unit length of linear probe, by N_A the average number per unit area of plane, and by N_V the average number per unit volume of convex field. When this convex field is transparent, we require its volume V_F and surface area S_F.

2.4 Product Integration

In developing numerical methods for the direct evaluation of inversion formulas such as (1.6a) and (1.6b), it is necessary to evaluate integrals of the form

$$I(F) = \int_x^X F(y)dy = \int_x^X w(y)f(y)dy = I(w;f) \text{ , } 0 \leq x \leq X \leq \infty \text{ ,} \tag{2.1}$$

where the notation allows for the possibility that the integrand $F(y)$ can be factorized into a smooth part $f(y)$ and a non-smooth (ill-behaved) part $w(y)$. In the application of a standard numerical quadrature procedure to $I(F)$, the strategy is to approximate $F(y)$ by some member of a family of smooth functions $\{F_n(y)\}$ (which depend on some parameter n) all of which can be integrated analytically. The most commonly used family is the polynomials of degree n which interpolate $F(y)$ at $n + 1$ points on $[x,X]$, viz. $\{p_n(F;y)\}$.

This strategy is implemented independently of whether or not $F(y)$ possesses a factorization into a product of a smooth and non-smooth component. Though convergence for Riemann integration can be proved for integrals which possess such a factorization, as long as the non-smooth component is itself integrable, this does not guarantee the convergence of more sophisticated quadrature rules than Riemann integration. See, for example, Elliott (1977) and the historical references mentioned there. In addition, even if such a method, which does have guaranteed convergence for the integral of interest, is used, the efficiency associated with its implementation may be grossly inferior when compared with its use to evaluate an integral with a smooth integrand.

The resulting conclusion, which appears to have been initially identified and implemented by Young (1954), is that the strategy for the construction of numerical quadrature procedures for such integrals should take into account explicitly the fact that such factorizations of the integrand F exist. The aim is then to choose a factorization of $F(y)$ into smooth and non-smooth parts, $f(y)$ and $w(y)$, respectively, in such a way that the chosen family of smooth functions $\{f_n(y)\}$, from which an approximation to f is taken, are such that the integrals

$$I(w;f_n(y)) = \int_x^X w(y)f_n(y)dy \tag{2.2}$$

can be evaluated analytically, thereby integrating out the non-smooth component

w(x) exactly. The more common choice of the $\{f_n(y)\}$ are linear combinations of basis (coordinate) functions $\{\phi_i(y)\}$, viz.

$$f_n(y) = \sum_{i=1}^{n} a_i^{(n)} \phi_i(y) , \tag{2.3}$$

such that the integrals

$$I(w;\phi_i(y)) = \int_x^X w(y)\phi_i(y)dy \tag{2.4}$$

can be evaluated analytically. Thus, for a given F(y), after an appropriate choice of w(y) and $\{\phi_i(y)\}$ have been made such that the integrals (2.4) can be evaluated analytically, the problem of product integration is reduced to an examination of appropriate facets of approximation theory concerning the approximation of some smooth f(x) by functions $f_n(y)$ of the form (2.3).

The justification for introducing product integration as the strategy for evaluating integrals of the mentioned type can be based not only on its success as a strategy for solving singular (integrable) integral equations (see, for example Weiss and Anderssen (1972), Weiss (1972), and Anderssen, de Hoog and Weiss (1973)) as well as evaluating singular (integrable) integrals (see, for example, de Hoog and Weiss (1973) and Atkinson (1975), Part I, Chapter 5), but also on the fact that, in statistical contexts such as the evaluation of functionals on particle size distribution, product integration estimators sometimes have superior properties when compared with the standard statistical estimators (see, for example, Anderssen and Jakeman (1975a)).

2.5 Spectral Differentiation

As we shall see below, the key to obtaining stable computational strategies for the direct evaluation of inversion formulas (such as (1.6a) and 1.6b)) on discrete data is a stable procedure for the numerical differentiation of observational data, which can be formulated as follows: *for given observational data*

$$d_j = s(x_j) + \varepsilon_j , \ 0 = x_0 < x_1 < x_2 \cdots < x_n = 1, \tag{2.5}$$

solve

$$s(x) = \int_0^1 H(x - y)\, u(y)\, dy = \underset{\sim}{L}u , \text{ say} , \tag{2.6}$$

where H(z) *denotes the Heaviside unit step function.* Because numerical differentiation is improperly posed in that small perturbations in the data can yield large perturbations in the solution, standard procedures based on an abstract formalism (such as finite difference methods) are inappropriate in that they function independently of the signal/noise structure in the observational data (2.5) to which they are applied. It is therefore necessary to appeal to sophisticated concepts and tools to obtain stable procedures which do allow a separation of the signal from the noise in observational data to be differentiated.

Two possible strategies are:

2.5.1 Regularization {Cullum(1971), Anderssen and Bloomfield(1974b) and Anderssen(1977)} For the observational data (2.5), estimate the derivative of the signal $s(x)$, namely $\dot{s}(x)$, as

$$u(\alpha;x_k),\ k = 0,1,2,\ \ldots,\ K,\ 0 = x_0 < x_1 < x_2 < \cdots < x_K = 1, \tag{2.7}$$

where $u(\alpha;x)$ is the function, contained in the Sobolev space $W_2^{(q)}$, for which

$$\min_{u \in W_2^{(q)}} \left\{ ||\underset{\sim}{L}u - s||_0^2 + \alpha\, ||u||_q^2 \right\}, \tag{2.8}$$

when evaluated on the data (2.5), attains its minimum, where $||\cdot||_q$ denotes the norm in $W_2^{(q)}$, namely

$$||u||_q^2 = ||u||_0^2 + ||u^{(1)}||_0^2 + \cdots + ||u^{(q)}||_0^2$$

with

$$u^{(\ell)} = d^\ell u/dx^\ell,\ \ell = 1,\ 2,\ \ldots,\ q,$$

and

$$||u||_0^2 = \int_0^1 u^2\,dx.$$

2.5.2 Wiener Filtering {Hannan (1970), Anderssen and Bloomfield (1974a) (1974b), Koopmans (1974) and Bloomfield (1976)} Assuming the observational data (2.5) has been generated by stationary stochastic processes, estimate $\dot{s}(x)$ as

$$\left.\begin{aligned} &u(\hat{\ell}(w)\ ;\ x_k) = \sum_{-\pi<w_j\le\pi} (iNw_j)\ \exp\ (ikw_j)\ \hat{\ell}\ (w_j)\ \tilde{v}\ (w_j), \\ &k = 0,1,2,\ \ldots,\ K,\ x_k = (k-1)\Delta,\ \Delta = 1/K,\ i = \sqrt{-1}, \end{aligned}\right\} \tag{2.9}$$

where $\hat{\ell}\ (w)$ is the chosen approximation for the *spectral window*

$$\ell(w) = g_s(w)\ /\ (g_s(w) + g_\varepsilon\ (w)) \tag{2.10}$$

with $g_s(w)$ and $g_\varepsilon(w)$ denoting the *spectral density functions* of $s(x)$ and $\{\varepsilon_j\}$, respectively, where

$$\tilde{v}(w_j) = (K + 1)^{-1} \sum_{k=0}^{K} d_k\ \exp\ (-\ ikw_j)$$

defines the *finite Fourier coefficients* of the data, and where $w_j = 2\pi j/K$. #

Computationally, both suffer from the same defect. The former will only yield a satisfactory approximation when an appropriate value has been estimated for α from the data, while, for the latter, an appropriate estimate $\hat{\ell}(w)$ of $\ell(w)$ is required. In both situations, estimates of the relevant unknown can be evaluated, if some appropriate criterion, which reflects the signal/noise structure in the given data (2.5), is used. Wahba (1976), (1977) has investigated, in a quite general context, the use of cross-validation for estimating α, while the statistical and time series literature contains numerous non-parametric approximations $\hat{\ell}(w)$ for $\ell(w)$ {see, for example, Jenkins and Watts (1968) and Bloomfield (1976)}.

On the other hand, Anderssen and Bloomfield (1974a,b) have proposed a parametric model for $\hat{\ell}(w)$. It is based on the fact that, under appropriate conditions (including $d_0 = d_K = 0$) and stationarity assumptions about $\{d_j\}$, if $\hat{\ell}(w)$ is modelled as

$$\lambda_\alpha^{(q)}(w) = [1 + \alpha(w/\Delta)^2 \sum_{j=0}^{q} (w/\Delta)^{2j}]^{-1}, \tag{2.11}$$

then

$$u(\alpha;x_k) = u(\hat{\ell}(w);\, x_k)\ , \; k = 1,\, 2,\, \ldots,\, K. \tag{2.12}$$

Not only does this equivalence yield a basis for explaining why the regularization (2.8) works, it also yields (by assuming that it is something special and therefore should be utilized), in conjunction with the work of Whittle(1952) on estimating parametric models for $g_S(w)$, a basis for estimating α with a specific optimal sense. In fact, if it is assumed that the errors $\{\varepsilon_j\}$ are *uncorrelated with mean zero and variance* $2\pi b$ *so that* $g_\varepsilon(w) = b$, then the parametric model for $g_s(w)$ becomes

$$g_s(w) \approx b\lambda_\alpha^{(q)}(w)/(1 - \lambda_\alpha^{(q)}(w)). \tag{2.13}$$

Since the distribution of a stochastic process is determined by its spectrum, at least for Gaussian data, α can be estimated from the statistical properties of the data $\{d_j\}$. In fact, for fixed q, Anderssen and Bloomfield (1974a) showed that, using Whittle (1952), the evaluation of a "maximum likelihood" estimate for α reduces to

$$\min_\alpha \Big[(K+1)/2 \log\Big\{\sum_{-\pi<w_j\leq\pi} I(w_j)\ [1 + \{\Big(\lambda_\alpha^{(q)}(w)\Big)^{-1} - 1\}^{-1}]^{-1}\Big\} + \sum_{-\pi<w_j\leq\pi} \log\,[1 + \{\Big(\lambda_\alpha^{(q)}(w)\Big)^{-1} - 1\}^{-1}]\ \Big] \tag{2.14}$$

where $I(w_j)$ denotes the *periodogram* defined by

$$I(w_j) = \frac{1}{2\pi(K+1)} \Big| \sum_{k=0}^{K} d_k \exp(ikw_j) \Big|^2 .$$

When α is estimated from (2.14), $\hat{\ell}(W) = \lambda_{\hat{\alpha}}^{(q)}(W)$, and $\hat{s}(x)$ is estimated as $u(\hat{\ell}(w)\ ;\ x_k)$, we refer to this estimate as the *(maximum likelihood) spectral derivative.*

§3. Sources for Abel-type Integral Equations in Application

When making an examination of the occurrence of some special equation, such as Abel-type integral equations, in applications, the usual practice is to list various disciplines such as physics, biology, seismology, medicine, geometric probability etc, and record for each listed discipline the different applications of the special equation. Such a classification for Abel-type integral equations is given in Table 1 {see Appendix - Table 1}. It is not as informative mathematically as it first looks. It clearly indicates that Abel-type integral equations often arise in applications, but gives no insight into why this happens. That is, it gives no insight into why, for some given application, it is an Abel-type equation which arises as the mathematical formalism rather than some other type of integral equation or even a differential equation.

On the other hand, the various applications of Abel-type integral equations could be classified on the basis of the underlying mathematical rationale for its occurrence. This is the motivation for the classification of Table 2 {see Appendix - Table 2}. What now becomes clear, and therefore is the justification for the classification presented in Table 2, are the reasons why Abel-type integral equations arise and will continue to arise so often in applications. The principal ones, namely rationales 1, 4, 6 and 8, are either directly or indirectly related to properties of axially and/or radially symmetric systems. Thus, the rationales 1, 4, 6 and 8 cannot be viewed as being completely independent. In fact, there is a close connection between 1 and 6. This relates to the way in which a uniformly random plane which sections the spherical particle P_X is defined. It depends on the initial choice of a random direction followed by the choice of a random point on the orthogonal projection of P_X with respect to the chosen direction - see Watson (1971), §2, and Miles and Davy (1977a). The geometric properties of the sections of the sphere of 1 are similar mathematically to the corresponding properties of the paths of projected rays through the cylinders and spheres of 6.

Clearly, the rationales 2, 3, 5 and 7 are completely independent of axial and radial symmetric considerations. In 3, it is the formal structure of the relationship between potential and kinetic energy which determines the Abel-type equation structure, while in 5 it is the nature of the kernel in Cauchy integral equations. In 7, it is the fact that the variance of a Brownian motion process is proportional to time of travel. As for 2, the explanation is directly connected with the existence of inversion formulas for separable Abel-type integral equations. This point is basic to much of the discussion in the sequel.

Even in an hour-long talk, time does not permit one to discuss in detail all these applications. We therefore restrict attention to the specific applications which arise in geometric probability. The justification for this is three-fold. The least known applications of Abel-type integral equations, but potentially some of the most important, arise in geometric probability. Lucid discussions of applications in areas other than geometric probability are better known and more readily available (for example Lonseth's (1977) discussion of sources and application of integral equations and the references listed in Tables 1 and 2). The geometric probability framework naturally yields applications of Abel-type integral equations with kernels of the form

$$k_1(y)k_2(x) \;/\; (x-y)^\alpha, \quad x \geq y \geq 0, \; 0 < \alpha < 1, \; \alpha \neq \tfrac{1}{2}.$$

Even though this last fact has no mathematical significance for the study of the various generalizations of Abel's original equation which have been proposed, it does represent positive independent motivation for studying the generalizations associated with the above mentioned kernels rather than some other generalization.

We therefore turn to an examination of the (mathematical) rationale behind the generation of Abel-type equations in geometric probability. Within the framework of geometric probability, it is that aspect of this subject which is

connected with (Underwood, de Wit and Moore (1976), p.v)

"the exploration of three-dimensional space (structure), when only two- (or one-) dimensional sections through solid bodies or their projection on a surface (or a line) are available.",

which is the major source for Abel-type integral equations in applications. This is clearly illustrated in Table 2. The inter-disciplinary activity associated with formulating theory and implementing (computational) methods to achieve the above goal (viz., *extrapolation from discrete (noisy) observations in lower dimensions* (one (lineal) or two (areal)) *to structure in higher* (three)) is now referred to as *stereology*. In fact, there is an international society called, not surprisingly, the International Stereological Society (ISS), which aims to stimulate communication between the different researchers concerned with such problems, whether their interest be mathematical or non-mathematical, theoretical or pragmatic, computational or qualitative, experimental or instrumental. Though the Foundation Meeting of the Society was only held in 1961, the existence of the Society has helped pull together into a limited number of sources basic information about stereology, its theoretical and experimental foundations, its applications, and its instrumentation. For those interested in pursuing this point further, the following sources are suggested:

1. W.L. Nicholson (Editor), Proceedings of the Symposium on Statistical and Probabilistic Problems in Metallurgy, Seattle, Washington, 4-6 August, 1971, in Special Supplement to Adv. Appl. Prob., December, 1972.

   A series of articles which illustrate the theoretical and experimental foundations, the applications and the instrumentation of stereology.

2. E.E. Underwood, R. de Wit and G.A. Moore (Editors) Fourth International Congress for Stereology, National Bureau of Standards Special Publication #431, U.S. Government Printing Office, Washington, Jan, 1976.

   Proceedings of the Fourth International meeting of the International Society for Stereology which was held at the National Bureau of Standards, Gaithersburg, Maryland, U.S.A. on September 4-9, 1975.

   A number of invited as well as contributed papers on all aspects of stereology.

3. E.E. Underwood, Quantitative Stereology, Addison-Wesley, Mass., 1970.

   A text book discussion of the subject along with a quite comprehensive set of references to applications.

4. R.T. De Hoff and R.N. Rhines, Quantitative Microscopy, McGraw-Hill, New York, 1968.

   Contains a series of invited articles by various experts in the field. A collection of articles on some of the theoretical and experimental foundations as well as the applications and instrumentation.

5. Journal of Microscopy which is the official Journal of the International Stereological Society.

6. Publications of Proceedings of the meetings of the International Stereological Society previous to that cited in 2. above. Further details from Prof. Anna-Mary Carpenter, University of Minnesota, Department of Anatomy, Minneapolis, Minn. 55455, U.S.A.

7. Other references cited in this paper such as the papers mentioned in §2.2.

Specifically, we examine the problem of estimating the size distribution of three-dimensional convex particles $\{P_x\}$ in an opaque medium from lower dimensional observations of them. The formalisms generated essentially fall into three categories related directly to the nature of the lower-dimensional data (Santaló (1955) and Jakeman and Anderssen (1975), §2 and Table 1)). The three categories, as defined by the lower dimensional data, are:

(a) intersections of convex particles by random plane sections taken through an opaque field;

(b) intersections of flat (no thickness) convex particles by random plane sections taken through an opaque field;

(c) intersections of convex particles with random linear probes taken through an opaque field.

The key is this. If we let $K(a,x)da$ be the probability that a particle P_x, when sectioned or probed randomly, has an area or length of intersection which takes a value between a and $a+da$, then $g(x)$ and $z(a)$ are related by the first kind Volterra integral equation

$$\int_{f(a)}^{\infty} x^p K(a,x)g(x)dx = c.z(a), \tag{3.1}$$

where the parameters $f(a)$, p and c are defined for the three categories (a), (b) and (c) above in Table 3. For example, for convex particles, Santaló (1955) obtains (3.1) by equating two different expressions for the average number of particle areas in the range $(a,a+da)$, which lie on random sectioning planes. One is defined in terms of $K(a,x)$ while the other is defined in terms of $z(a)$.

TABLE 3

Table for the Parameters $f(a)$, p and c of Equation (3.1)

(the constants C, a_n, N_A, N_V, etc are defined in §2)

Data Category	f(a)	p	c
(a)	$(a/a_n)^{1/2}$	1	$2\pi N_A/(CN_V)$
(b)	a/a_n	1	$4N_A/(LN_V)$
(c)	a/a_n	2	$4N_L/(SN_V)$

Thus, the problem of generating the various formalisms, reduces to determining explicitly the structure of $K(a,x)$. In Table 4 {see Appendix - Table 4}, various forms taken by $K(a,x)$ and the resulting forms for (3.1) are listed along with the associated size distribution estimation problem. Even though the two applications associated with (c) correspond to (numerical) differentiation, they are included because (numerical) differentiation corresponds to the degenerate situation where $\mu = 0$ and $k(y,x)$ separable in (1.2).

For example, consider the size distribution estimation problem for circular sections observed on random plane sections taken through an opaque field of spheres. From the point of view of estimating $K(a,x)\,da$, the key property of similarly shaped convex particles is that "an area $\underline{a}$ on a planar section through the particle P_1 corresponds to an area $x^2 a$ on a planar section through the particle P_x". Consequently, we obtain

$$K(a, 1)da = K(ax^2, x)\ d\ (x^2a)$$

and hence, on writing $\alpha = ax^2$,

$$K(\alpha, x)d\alpha = K(\alpha/x^2, 1)\ d\alpha/x^2. \tag{3.2}$$

However, for a sphere of radius R, since "the probability the sectioned circle has a radius in the range r to $r + dr$" equals "the probability the sphere is cut at a distance from its centre in the range x to $x + dx$", which in turn equals "xdx (uniformly distributed)", it follows that

$$K(a, 1) = \left(4\pi(\pi R^2 - a)\right)^{-\frac{1}{2}} = \left(4\pi(a_n - a)\right)^{-\frac{1}{2}}$$

which yields

$$K(a, p) = (4\pi p^2(a_n p^2 - a))^{-\frac{1}{2}}\ ,\ p = x/R, \tag{3.3}$$

with the largest observed particle scaled to correspond to $p = 1$.

Thus, for spheres, (3.1) becomes, on using (3.3) and the appropriate results of Table 3,

$$\int_{(a/a_n)^{\frac{1}{2}}}^{\infty} \frac{g(x)dx}{2(\pi(a_n x^2-a))^{\frac{1}{2}}} = \frac{2\pi N_A}{CN_V}\ ,$$

which in turn reduces to

$$\int_{y/y_n}^{\infty} \frac{g(x)dx}{(y_n{}^2x^2-y^2)^{\frac{1}{2}}} = \frac{m}{y}\,h(y) \tag{3.4}$$

since, for spheres, $C = 4\pi$, $N_A/N_V = 2m = \int_0^\infty x g(x)\,dx$, $a_n = \pi y_n^2$, $a = \pi y^2$,

$z(a)da = h(y)dy$ with $z(a) = h(y)/2\pi y$, where $h(y)$ denotes the probability density for the radii of the spheres. On introducing the substitution $u = xy_n$, the first kind Abel-type equation known as the random spheres model is obtained:

$$\int_y^\infty \frac{g(u)du}{(u^2-y^2)^{\frac{1}{2}}} = \frac{m}{y}\,h(y) \tag{3.5}$$

§4. The Numerical Analysis of Abel-type Integral Equations

The foundations of the numerical analysis of any special class of equations (such as Abel-type integral equations) rests heavily on the known results of mathematical analysis about the properties of such equations. Such properties include existence, uniqueness and smoothness conditions for their solutions, conditions under which closed form solutions and inversion formulas can be constructed, existence of different (but equivalent) representations for their solutions, existence of families of complete functions for which the inversion of subclasses of the given equations can be constructed analytically, and the properties of discretized versions of the given equations. For the separable Abel-type integral equations, they not only form the most important subclass of Abel-type from the point of view of the regularity with which they arise in applications, but also in terms of the major extent to which properties of the above type are known.

From the point of view of the present discussion, the important properties of separable Abel-type equations are:

(i) *Existence, Uniqueness and Smoothness*. As already mentioned in the Preliminaries, the source reference is Atkinson (1974).#

(ii) *Weakly Improperly Posedness*. Because the numerical differentiation of discrete observational data is improperly posed in the sense that small perturbations in the data can yield large perturbations in the solution, the inversion of first-kind Abel-type equations for discrete observational data is regarded to be weakly improperly posed, since it corresponds to a fractional differentiation of the data. Perturbations $\delta u(x)$ in the solution of the Abel equation (T1.4) of Table 1, with $k_1(y) = y/m$ and $k_2(x,y) = 1$, with $m = \int_0^a xu(x)dx$, corresponding to different perturbations $\delta s(y)$ in the data are illustrated in Table 5.#

(iii) *Existence of Inversion Formulas*. Along with appropriate assumptions, various extensions and generalizations of the correspondence between the Abel-type equation (1.5) and the inversion formulas (1.6a) and (1.6b) are known. For example, when (1.5) is replaced by

$$y^{-\beta}k_1(y)\int_0^y \frac{k_2(x)u(x)}{(y^p - x^p)^\alpha}\,dx = s(y)\ ,\ 0 < y \leq Y\ ,\ p\alpha - \beta > 0\ , \tag{4.1}$$

with $k_1(z) \neq 0$ and $k_2(z) \neq 0$, $0 \leq z \leq Y$, the inversion formula

$$u(x) = \frac{x^{p\alpha+\beta-1}}{k_2(x)}\ [a + x\,\phi(x)]$$

with

$$a = \frac{p\ \sin\alpha\pi}{\pi}\ (p\alpha + \beta)\left[\frac{s(0)}{k_1(0)}\right]\int_0^1 \frac{z^{p+\beta-1}}{(1-z^p)^{1-\alpha}}\,dz\ ,$$

and

$$\phi(x) = \frac{p\ \sin\alpha\pi}{\pi}\int_0^1 \frac{z^{p+\beta}}{(1-z^p)^{1-\alpha}}\left[\frac{d}{d(zx)}\left(\frac{s(zx)}{k_1(zx)}\right) + \frac{p\alpha+\beta}{zx}\left(\frac{s(zx)}{k_1(zx)} - \frac{s(0)}{k_1(0)}\right]\right]dz\ ,$$

TABLE 5

Exemplification of the Weakly Improperly Posed Nature of First Kind Abel-type Integral Equations

$\delta s(y)$	$\delta u(x)$
1. $\frac{\Gamma(\nu)\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})}\, y(a^2-y^2)^{\nu-\frac{1}{2}}/m$	$2y(a^2-x^2)^{\nu-1}$, Re $\nu > 0$
Special Cases	
$\nu = \frac{1}{2}$ $\quad \pi y/m$ ($s(y)$ bounded on $0 \leq y \leq a$)	$2x(a^2-x^2)^{-\frac{1}{2}}$ ($u(x)$ unbounded on $0 \leq x \leq a$)
$\nu = 1$ $\quad \pi y(a^2-y^2)^{\frac{1}{2}}/m$ ($\lim_{y\to a} s(y) = 0$)	$2x$ ($\lim_{y\to a} u(x) = 2a$)
2. $\pi y \left\{ {\cos \atop \sin} \right\} (\frac{1}{2}\alpha t)\ \ J_0(\frac{1}{2}\alpha t)/m$ ($t = a^2-y^2$) (bounded on $0 \leq y \leq a$)	$2x(a^2-x^2)^{-\frac{1}{2}} \left\{ {\cos \atop \sin} \right\} (\alpha(a^2-x^2))$ (unbounded on $0 \leq x \leq a$)
3. $\frac{\Gamma(\nu)\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} \times t^{\nu-\frac{1}{2}}\ {}_1F_1(\nu;\ \nu+\frac{1}{2};\ \alpha t)/m$ ($t = a^2-y^2$)	$2x(a^2-x^2)^{\nu-1}\ \exp(\alpha(a^2-x^2))$, Re $\nu > 0$

can be derived from equations (2.1), (2.2) and (2.5) of Atkinson (1974). In fact, as long as conditions are satisfied which guarantee existence, uniqueness and appropriate smoothness of their solutions, all first kind Abel-type integral equations with separable kernels have inversion formulas.#

(iv) *Existence of Complete Systems for which the Inversion is Known Analytically.*

Let

$$\underset{\sim}{A}u = s\ ,\ \underset{\sim}{A} : \underline{\underline{S}}_1 \to \underline{\underline{S}}_2\ , \tag{4.2}$$

where $\underline{\underline{S}}_1$ and $\underline{\underline{S}}_2$ are given functional spaces, denote a densely defined operator representation for some given first kind Abel-type integral equation, and let $\{\psi_k(x)\}$ denote a explicitly-known coordinate system which spans the domain of $\underset{\sim}{A}$, $\underline{\underline{D}}(\underset{\sim}{A})$, such that the system $\{\phi_k = \underset{\sim}{A}\psi\}$ is also known explicitly and spans the range of $\underset{\sim}{A}$, $\underline{\underline{R}}(A)$, dense in $\underline{\underline{S}}_2$. For such systems, if

$$s = \sum_{k=1}^{n} a_k^{(n)}\ \ \phi_k\ ,\ n < \infty\ ,$$

then the inversion can be performed analytically to yield

$$u = \sum_{k=1}^{n} a_k^{(n)}\ \ \psi_k\ ,\ n < \infty\ ,$$

TABLE 6

Some Invertible Coordinate Systems* for the First Kind Abel-type Integral Equation $s(y) = 2\int_y^a \frac{xu(x)}{(x^2-y^2)^{1/2}}\,dx\ ,\ 0 \le y \le x \le a \le \infty.$

	$\{\phi_k\}$	$\{\psi_k\}$	Source Reference
1.	$\phi_0(y) = (a^2-y^2)^{1/2}$, $\phi_k(y) = (a^2-y^2)^{k-1},\ k \ge 1.$	$\psi_0(x) = 1/\pi$ $\psi_k(x) = \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k)\Gamma(\frac{1}{2})}(a^2-x^2)^{k-\frac{1}{2}},\ k \ge 1$	Minerbo and Levy (1969).
2.	$\phi_k(y) = t^{\beta}T_k(1-2t),\ k \ge 0,$ $T_k(z)$ Chebyshev polynomial of degree k , $t = a^2-y^2$, β = const. $> -\frac{1}{2}$.	$\psi_k(x) = \frac{t^{\mu+\beta-1}\Gamma(1+\beta)}{\Gamma(\mu+\beta)\Gamma(1-\mu)} \times$ $\times\,{}_3F_2(-k,k,\beta+1;\frac{1}{2}\ ,\ \mu+\beta;y),$ $t = a^2 - x^2$	Piessens and Verbaeten (1973).
3.	$\phi_k(y) = f_k(a^2-y^2)$	$\psi_k(x) = \Gamma(\mu)\ F_k\ (a^2-x^2)$	Erdelyi(1954)

where $f_k(x)$ and $F_k(y)$ corresponding to any systems of functions in a table of Rieman-Liouville integral transforms:

$$F_k(y) = [\Gamma(\mu)]^{-1}\int_0^y f_k(x)\ (y-x)^{\mu-1}\ dx.$$

* Note 1. The corresponding invertible coordinate systems for other first kind Abel-type integral equations can be obtained by transformation.

** Note 2. Similar systems exist for the equation $s(y) = \frac{y}{m}\int_y^{\infty} \frac{u(x)}{(x^2-y^2)^{1/2}}\,dx.$

The obvious source is a table of Weyl integral transforms - Erdelyi (1954).

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when the mapping $\underset{\sim}{A} : \underline{D}\ (\underset{\sim}{A}) \to \underline{R}\ (\underset{\sim}{A})$ is one-to-one. Because of the indirect connection with variational and projection methods, we shall refer to the dual system $\{\psi_k\ ,\ \phi_k = \underset{\sim}{A}\psi_k\}$ as an $\underset{\sim}{A}$- *invertible coordinate system.*

Table 6 contains some examples of $\underset{\sim}{A}$-invertible coordinate systems for some commonly occurring Abel-type integral equations. It is pertinent to note that such explicit systems can only be constructed for Abel-type equations for which the kernel $K(y,x)$ of (1.2) is known explicitly. The existence of such systems is basic to the use of pseudo-analytic methods which will be discussed in some detail below.#

In part, the existence of this extensive information base about the mathematical properties of separable Abel-type integral equations explains why the literature on the numerical analysis of such equations is not only extensive but also contains a great variety of mathematically independent proposals for their solution numerically. They include the standard finite difference approaches of Linz (1968), Brunner (1973), (1974), product integration procedures of Young (1954), Linz (1967), Noble (1970), Weiss (1972), Weiss and Anderssen (1972), and Anderssen, de Hoog and Weiss (1973), the pseudo-analytic methods of Minerbo and Levy (1969) and Piessens and Verbaeten (1973), use of finite difference methods by Nestor and Olsen (1960), to evaluate the inversion formulas and the proposal by Anderssen (1973), (1976) to use the spectral differentiation procedure of Anderssen and Bloomfield (1974a,b) to evaluate the inversion formulas and its implementation by Anderssen and Jakeman (1975a,b) for Abel-type equations in stereology. Except for the recent paper of Baev and Glasko (1976), the direct application of regularization does not appear to have been examined.

Most of the numerical methods proposed in the stereological literature for the approximate solution of Abel-type equations correspond to the application of (standard) finite difference techniques (such as ones based on the use of rectangular quadrature (Watson (1971)) and variants of product mid-point (Wicksell (1925), Meisner (1967), Baudhuin and Berthet (1967), and Weibel *et al* (1969)) in conjunction with a data grouping strategy to obtain a discrete approximation to the size distribution data $s(y)$ from given observations. Few appear to have known that an inversion formula existed and even fewer used it (Meisner (1967), Appendix, makes indirect use of it when deriving equations (A.12)). Some, as cited above, appear to have used product integration techniques without explicitly acknowledging that this was the situation. For example, the formulas on p.101 of Tallis (1970) correspond to the use of a product-trapezoidal technique, while those of (13) of Meisner (1967) correspond to product midpoint. Few appear to have explored the possibility of working directly with the cumulative distribution generated by the observational data. A number have proposed methods of a more specialized nature such as the parameter estimation procedure of Keiding *et al* (1972) and the successive subtraction algorithm of Saltikov (1967); but substantive justification for using them does not appear to exist, while strong reasons for not using them have been put forward by Jakeman (1975), §3.3, and Anderssen and Jakeman (1975).

Another key reason for the variety of such proposals is the existence of applications where the data $s(y)$ is known analytically, such as the calculation of boundary crossing probabilities for Brownian motion and the associated applications mentioned in Table 1. In fact, many of the methods for Abel-type equations, which do not involve any form of stabilization, can be applied successfully to the solution of this subclass of problems, if, *with respect to the chosen discretization of the given Abel-type equation, the required discretization of* $s(y)$ *is computed "sufficiently" accurately*. However, space, time and context does not permit even a cursory discussion of all aspects of the numerical analysis of Abel-type equations, and thus the exact data subclass, in particular. The interested reader is referred to Weiss(1972), and Anderssen, de Hoog, and Weiss (1973), as well as the more recent work of Holyhead and McKee (1976), (1977).

We therefore limit attention to the numerical solution of first kind Abel-type equations of the form

$$s(y) = \int_y^Y k_1(y)\, k_2(x)\, (x - y)^{-\mu} u(x)\, dx, \quad 0 \leqslant \mu < 1, \quad 0 \leqslant y \leqslant x \leqslant Y < \infty, \tag{4.3}$$

where μ, Y, $k_1(y)$ and $k_2(x)$ are known explicitly, and when the data $s(y)$ is only available as discrete observations:

$$\{d_i\} = \{d_i = s(x_i) + \varepsilon_i,\ i = 1, 2, \ldots, n;\ \varepsilon_i \text{ discrete random errors}\}. \tag{4.4}$$

From the point of view of proposing a numerical procedure for the approximate solution of (4.3) for the given data $\{d_i\}$, the aim is to obtain from the given data as much information as possible by maximizing knowledge about the properties of the solution of (4.3). It is now necessary to be sympathetic with the "red Queen's philosophy"

> *"I could have done it in a much more complicated way", said the red Queen, immensely proud.*
>
> *Lewis Carroll*

in that, in trying to achieve the above goal, it may be necessary to be devious and complicated as well as crafty.

By themselves, parsimony and simplicity should not now be the guiding strategy. Instead, a suitable trade-off between the competing claims must be made and implemented. For the solution of first kind Abel-type equations, the competing claims are:

(a) smoothing of the data (separation of the errors (noise) from the structure (signal) in the data);

(b) construction of a stabilized computational strategy for the solution of the formalism (4.3).

Many of the methods proposed fail to do either, but these are usually the methods proposed for exact data, whereas the above claims are first and foremost a direct consequence of the assumption that the data is discrete and observational. Additional motivation for (b) is the previously mentioned weakly improperly posed nature of first kind Abel-type equations.

This weakly improperly posed nature is also the reason for the failure of standard finite difference methods for the solution of (4.3) when the data is discrete and observational (or discrete, but not computed with sufficient accuracy). As shown in Anderssen (1976), §2, they have no natural stabilization against the propagation of small errors in the data into unacceptably large errors in the solution. In fact, finite difference methods fail to meet either of the claims (a) or (b). Insight into why this is so can be gained from an examination of the use of finite difference methods for the differentiation of discrete observational data, since the solution of Abel-type equations corresponds to a fractional differentiation of the data.

There are two obvious ways by which the above competing claims can be satisfied. They are:

1. Apply either the inversion formula (1.6a) or (1.6b) to filtered data. The simplest implementations of this strategy are the *pseudo-analytic methods* in which the data $\{d_i\}$ is smoothed using the model

$$S_n(y) = \sum_{k=1}^{n} a_k^{(n)} \phi_k(y) \ , \ n < \infty, \tag{4.5}$$

and an appropriate statistical fitting strategy, such as least squares, where $\{\phi_k(y)\}$ belongs to an explicitly known $\underset{\sim}{A}$-invertible coordinate system $\{\psi_k, \phi_k\}$. Consequently, the application of the inversion formula is accomplished analytically to yield the approximation

$$u_n(x) = \sum_{k=1}^{n} a_k^{(n)} \psi_k(x). \tag{4.6}$$

2. Solve the given first kind Abel-type equation using a stabilized computational strategy such as regularization and, if necessary, filter (smooth) the results. Except for the work of Baev and Glasko (1976), little appears to have been done with this approach, though it has potential when viewed as the basis for a direct filtering strategy in the sense of spectral differentiation.

A third, but not so obvious way, is to perform the filtering and stabilization simultaneously. This is the basis of the methods proposed by Anderssen (1976) and studied in some detail by Jakeman (1975). They involve, through the use of product integration and spectral differentiation, the direct stabilization of the inversion formulas (1.6a) and (1.6b) so that they can be applied to the given discrete observational data $\{d_i\}$.

For reasons which will become clearer below, we shall limit attention to an examination of pseudo-analytic methods and the evaluation of stabilized inversion formulas.

Though they have obvious disadvantages such as

dis (i) The smooth representation $s_n(y)$ of (4.5) is constructed with a special sense (i.e., the form of $\{\phi_k\}$) which might not be representative of the true structure in the data.

dis (ii) Since (4.3) is weakly improperly posed, the use of the smooth representation $s_n(y)$ which does not truly approximate the structure in the data $\{d_i\}$ could yield incorrect results (e.g., see Figure 2 in Anderssen (1973)).

the pseudo-analytic methods do have the following advantages

ad (i) They are simple to implement - once a suitable $\{\psi_k, \phi_k\}$ has been chosen, it is only necessary to do linear parameter estimation with respect to an appropriate assumption about the structure of the errors in the data $\{d_i\}$.

ad (ii) They are not restricted to evenly spaced data.

ad (iii) Assuming that $s_n(y)$ of (4.5) is a reasonable approximation for the true structure in the data and the parameter estimation problem is numerically stable, then the totality of data points required for a satisfactory solution is potentially not too large.

On the other hand, though the evaluation of stabilized inversion formulas have obvious advantages which more or less correspond to the negation of *dis*(i) and *dis*(ii), it also has disadvantages, directly connected with the use of spectral differentiation, which more or less correspond to the negation of *ad*(i), *ad*(ii) and *ad*(iii).

Clearly, the key point is that they potentially define independent rather than related methods, and therefore should only be compared from the point of view of identifying their compatibility. For example, the pseudo-analytic methods can be viewed as simple strategies for generating first order approximations to the required solution independently of the signal structure in the data $\{d_i\}$, but not more accurate approximations; while the evaluation of the stabilized inversion formulas defines a sophisticated method for computing an accurate approximation to the required solution but only when the signal structure

in the data $\{d_i\}$ satisfies conditions compatible with such a method (i.e., with the use of spectral differentiation).

Thus, the sensible strategy would appear to be to use them as compatible methods so that their advantages support each other at the expense of their disadvantages. In particular, for given data $\{d_i\}$, generate a first approximation to the required solution using a pseudo-analytic method in such a way that the residual data has a form compatible with the assumptions which underlie the use of one of the stabilized inversion formulas. This is the basis for the computational strategy proposed by Anderssen (1976) and studied in some detail in Jakeman (1975)

More explicit details about the algorithm is given in "Appendix - The Algorithm".

§5. Linear Functionals and the Process of Inference

When theoretical and numerical investigations of special classes of equations (such as Abel-type integral equations) are conducted independently of applications, information about the structure of the inference processes, which use the solutions generated by such investigations and which are used to make conclusions about some given application, are ignored. For certain theoretical investigations (such as the study of the existence and smoothness of solutions), there may be justification for ignoring this information. But, as we shall show below for separable Abel-type equations in application, this information is of importance for some aspects of theoretical investigations (such as the construction of inversion formulas) and is of paramount importance for numerical investigations, as it may allow the numerical problem to be recast in a computationally more favourable framework.

A rationale and motivation for making wherever possible such a change of framework is: *Within a more favourable computational framework, the probability is higher that inferences consistent with the information in given discrete observational data will be generated.*

In fact, for separable Abel-type integral equations, when the inference process is based solely on the use of linear functionals of the solution, the mildly improperly posed problem, consisting of first solving the relevant Abel-type equation for discrete observational data followed by the evaluation of the linear functionals on the resulting approximate solution, can be replaced by the direct evaluation of the linear functionals on the given discrete observational data.

To illustrate, we consider the following problem: *for a given consignment of steel, determine whether it has the (impact) strength to do a given job.* As the result of extensive experimentation, metallurgists claim that (Hyam and Nutting (1956))

(i) *Steel consists of* a conglomerate of ferrous crystals and carbide (carbon-type) particles, which to a first approximation can be regarded as spheres.

(ii) *The strength of steel depends on its hardness.* If it is too hard, it is too brittle. Tempering is used to make it softer and thereby increase its strength.

(iii) *The hardness of tempered steels depend upon the ferrite grain sizes.* The general rule is "the larger the ferrite grains, the softer the steel".

(iv) *The growth of the ferrite grains is limited by the carbide particles.* The carbide particles form along grain boundaries, and therefore the size of the ferrite grains is heavily constrained by the size distribution and number density of the carbide particles (assuming the centers of the particles are distributed something like a Poisson process).

(v) *During tempering, the number density of carbide particles decreases but their sizes grow, so the ferrite grains grow.* As tempering progresses, the microstructure and mechanical properties continue to alter, due to the development of approximately spherical particles of carbide (cementite) which gradually grow in size and decrease in number.

The mentioned change in size and number of carbide particles can only occur by the solution of the smaller particles, diffusion of carbon through the ferrite, and the simultaneous growth of the larger particles. This indicates that, because the amount of ferrite, carbon, etc. in a given consignment of steel being tempered is fixed, there is *a trade-off between number density* ρ *and size distribution* $u(x)$ of carbide particles during tempering at the expense of hardness and hence at the gain of strength.

Thus, since a knowledge of the size distribution $u(x)$ is necessary for estimating the number density ρ, the problem is reduced to an examination of the properties of the size distribution $u(x)$. This completes the *first step* in the analysis of an application - *pinpoint the properties of the application which relate directly to the problem of interest.*

The *second step* is to derive a *mathematical formalism* which relates $u(x)$ to observable data. As a first approximation, we shall assume that the carbide particles are spherical. Since it is not possible to observe the carbide spheres and hence $u(x)$ directly, it is necessary to estimate it by some indirect procedure. It is because of this necessity to observe the thing one requires by some indirect procedure that improperly posed formulations often arise. The ways in which this can be done are not unique, and depend heavily on the sampling strategy used. The one which we shall pursue here is to use the size distribution of circles $s(y)$ (i.e., the radius or diameter distribution) defined on random plane sections taken through the steel. The sampling strategy is: on random plane (polished) sections taken through a suitably chosen sample of the steel, measure the observed size distribution of circles. This will yield the data $\{d_i\}$.

The motivation for assuming first that the carbide particles are spherical (as well as agreeing approximately with observation) and for taking the size distribution of circles $s(x)$, as the means by which we estimate $u(x)$, is the existence of the following mathematical formalism (an Abel-type integral equation) which relates $u(x)$ and $s(y)$:

$$s(y) = \frac{y}{m} \int_y^a u(x) \,/\, (x^2 - y^2)^{\frac{1}{2}} \, dx, \quad 0 \leq y \leq x \leq a < \infty, \tag{5.1}$$

where $[0,a]$ defines the support of $u(x)$ and

$$m = \int_0^a xu(x) \, dx \tag{5.2}$$

denotes the average sphere radius.

<u>Note 5.1</u>. The actual procedure, for choosing a sample of steel (from the consignment), and then taking random plane sections through it, is non-trivial and involves a number of deep (probabilistic) questions. A discussion of some of the difficulties which arise along with proposals for using volume weighted and area weighted sampling can be found in the references of §2.2.

The sampling and measurement of circles will yield discrete observations of the cumulative distribution

$$S(y) = \int_0^y s(z) \, dz \text{ , } s(0) = 0 \text{ , } S(a) = 1,$$

namely

$$d_i = S(y_i) + \varepsilon_i \ , \ 0 < y_1 < y_2 < \dots < y_n \ .$$

We assume that the measurement process is sufficiently accurate so that duplicate observations do not occur. Once n is sufficiently large (so that a sufficiently detailed amount of information about the structure of $S(y)$ has been accumulated), the *third step* can be implemented: apply a viable computational strategy to the data $\{d_i\}$ to generate an approximation

$$\{\hat{u}_j\} = \{\hat{u}(x_j);\ \ x_j = (j - 1)\ \Delta x,\ j = 1, 2, \dots, N,\ \Delta x = a/(N - 1),\ N < n\}$$

to $u(x)$ on an even grid $\{x_j\}$. For example, apply a two-stage procedure along the lines proposed by Anderssen (1976), implemented by Anderssen and Jakeman (1975b), and discussed in §4 above.

The *fourth step* involves the use of the approximation $\{\hat{u}_i\}$ of $u(x)$ to derive inferences about the strength of the steel in the given consignment. But, here is the catch. Though the metallurgist argues that the strength of steel is governed by the size distribution $u(x)$ of the carbide particles, he doesn't necessarily use $u(x)$ directly {see, for example, Hilliard (1968), p.1373, Underwood (1970), p.149, Hyam and Nutting (1956)} when making a decision about whether a given steel has the strength to do a given job. Instead, he makes inferences on the basis of the values of functionals defined on $u(x)$. The functionals of $u(x)$ commonly used by metallurgists for this purpose are (see, for example, Hyam and Nutting (1956)):

a. average sphere radius m of (5.2) (which we assumed above to be known);

b. average particle surface area

$$A_u = 4\pi \int_0^a x^2 u(x)dx \ ; \tag{5.3}$$

c. average particle volume

$$V_u = \frac{4\pi}{3} \int_0^a x^3 u(x)dx \ ; \tag{5.4}$$

d. number of particles N_V per unit volume (assuming their centers follow a Poisson distribution)

$$N_V = N_A \ / \ (2m) \ , \tag{5.5}$$

where N_A denotes the number of circles per unit area of planar section of the sample of steel on which the data $\{d_i\}$ was measured.

Thus, the implementation of *step four* initially involves the evaluation of the functionals m, A_u and V_u and whence N_V. Invariably, metallurgists estimate m, A_u and V_u by evaluating the integrals (5.2), (5.3) and (5.4) using the approximation $\{\hat{u}_i\}$ - see, for example, Hyam and Nutting (1956), p.151, column 2. They miss the point that, if only estimates of m, A_u, V_u and N_V are required for inference purposes, then the necessity to explicitly estimate $u(x)$ can be circumvented.

Because inversion formulas for separable first kind Abel-type integral equations are known explicitly, they can be used to replace linear functionals defined on $u(x)$, namely

$$F(\theta;z) = \int_z^a \theta(x)u(x)dx,$$

by linear functionals on $s(x)$, namely

$$F(\phi;z) = \int_z^a \phi(z;x)\ s(x)dx,$$

In fact, for a given $\theta(x)$, the $\phi(z;x)$ corresponding to the Abel integral equation (4.3), with $k_1(y) = y/m$ and $k_2(x) = 1$, is given by

$$\phi(z;x) = \frac{2m}{\pi}\{\theta(z)(x^2 - z^2)^{-\frac{1}{2}} + \int_z^x \frac{d\theta}{dy}(x^2 - y^2)^{-\frac{1}{2}}dy\}, \tag{5.6}$$

if we assume that $\theta(x)$ is such that $\frac{d\theta}{dy}(x^2 - y^2)^{-\frac{1}{2}}$ is integrable, and $\theta(y) \neq y$. When $\theta(y) = y$, it is necessary to employ the type of independent argument used by Watson (1971), since the above formula (5.6) for $\phi(x)$ involves m. The existence of the general correspondence (5.6) between $\theta(x)$ and $\phi(x)$ appear to have been overlooked in practice, except for the special cases examined by Meisner (1967), Watson (1971) and Jakeman (1975; eqn(4.3.1)).

Thus, the problem of estimating the functionals (5.2), (5.3), (5.4) and (5.5) reduces to evaluating the functionals

$$m = \frac{\pi}{2}\ (\int_0^a \frac{s(x)}{x}\,dx\ )^{-1} \tag{5.7}$$

$$A_u = 16m \int_0^a xs(x)dx \tag{5.8}$$

$$V_u = 2\pi m \int_0^a x^2 s(x)dx \tag{5.9}$$

$$N_V = N_A\ /\ (2m) \tag{5.10}$$

which are properly posed formulations. Though standard quadrature rules, such as the rectangular, trapezoidal and Simpson, can be applied to evaluate the above functionals for the data $\{d_i\}$, it doesn't follow that they are necessarily the most appropriate. Not only is it necessary to choose a quadrature which performs well numerically; it is also necessary, because of the observational nature of the data $\{d_i\}$ {see eqn(4.4)}, to choose quadrature rules which yield

estimators of the above functionals which have appropriate statistical properties. This problem has been examined in some detail by Anderssen and Jakeman (1975a) and Jakeman and Scheaffer (1977). They conclude that, especially for functionals involving integrals of the form

$$\int_z^a \frac{s(x)}{(x^2-z^2)^{\frac{1}{2}}}\, dx, \quad 0 \leq z < a,$$

product integration procedures should be used.

From a general point of view, the above discussion illustrates that the strategy for problem solving is not simply

(i) State (define) the problem.

(ii) Identify the appropriate (experimental, observational and/or theoretical) facts about the problem and its context from which one derives structural assumptions for the construction of a framework into which the problem can be embedded rigorously (mathematically, scientifically, logically or unambiguously).

(iii) Define a mathematical formalism for the problem, and, when necessary, an appropriate sampling strategy for the data.

(iv) Through the use of an appropriate rationale (consistent with the mathematical formalism), derive a viable computational formalism for the mathematical formalism.

(v) Construct an algorithmic implementation for the computational formalism, and evaluate a specific approximation to the required solution of the mathematical formalism.

(vi) Use the resulting approximation for inference purposes about the problem context.

In fact, the strategy should be (assuming backtracking takes place when and where necessary)

a. (i)

b. (ii)

c. (iii)

d. Identify the functionals, defined on the solution of the mathematical formalism, from which inferences about the problem will be derived.

e. Through the use of an appropriate rationale (consistent with the mathematical formalism and the identified functionals), derive a viable computational formalism for the mathematical formalism.

f. Construct an algorithmic implementation for the computational formalism, and evaluate a specific approximation to the required functionals.

g. Use the resulting estimates of the functionals for inference purposes.

APPENDIX - TABLE 1

Classification of Abel-type Integral Equations in Application on the Basis of Subject Area of Application.

Subject Area	Particular Form of the Abel-type Integral Equation	Context of the Application
1a. Chemistry and Physics - second phase copolymer spheroids dispersed in a continuous first phase of polystyrene. (Meisner (1967))	Equation (T1.1) $$s(y) = k_1(y) \int_y^\infty \frac{k_2(x,y)u(x)}{(x^2 - y^2)^{\frac{1}{2}}}\, dx,$$ $0 \leq y \leq x \leq \infty$, where $k_1(y)$ and $k_2(x,y)$ are known non-singular functions on $[0,\infty)$.	($k_1(y) = y/m$, $m = \int_0^\infty xu(x)dx$, $k_2(x,y) \equiv 1$) The determination of the impact value of high-impact polystyrene from the size distribution $u(x)$ of the copolymer spheroids, which in turn is determined from the size distribution of the spheroid sections on random planes taken through the polystrene.
1b. Geometric Probability - the random sphere {random ellipsoid of one parameter} model in stereology. (Wicksell (1925,1926), Moran (1972))		($k_1(y) = y/m$, $m = \int_0^\infty xu(x)dx$, $k_2(x,y) \equiv 1$) The determination of the size distribution of spheres {ellipsoids of one parameter} $u(x)$ in a three-dimensional opaque aggregate from observations of the size distribution of circles {ellipses of one parameter} $s(y)$ on random plane sections taken through the aggregate.
1c. Metallurgy - size distribution of Mn S (manganese sulphide) inclusions in free machining steel. (Hilliard and Riekels(1977))		($k_1(y) = y/m$, $m = \int_0^\infty xu(x)dx$, $k_2(x,y) \equiv 1$) The determination of the size distribution of Mn S inclusions when the observations of the size distribution of their sections on random planes are interpreted as a size distribution of circles $s(y)$ by taking the mean caliper diameter of the sections as the diameter of the circles. The size distribution of the Mn S inclusions thereby becomes a size distribution of spheres $u(x)$ defined by the equivalent circular sections.
1d. Metallurgy - the impact strength of plain carbon steels. (Hyam and Nutting(1956))		($k_1(y) = y/m$, $m = \int_0^\infty xu(x)dx$, $k_2(x,y) \equiv 1$) On the basis of known relationships between the impact strength of plain carbon steels and the size distribution of carbide particles within such steels, their impact strength can be determined once the size distribution of the carbide particles is known. Assuming the carbide particles are spherical, the context reduces to that mentioned above for the size distribution of Mn S inclusions.
1c. X-ray Scattering - slit correction in small-angle x-ray scattering. (Mazur (1971))		($k_1(y) = 2$, $k_2(x,y) = xW((x^2-y^2)^{\frac{1}{2}})$, W the (experimentally determined) slit weight function) Determine the true scattering intensity $u(x)$ from the observed scattering intensity $s(y)$.

<u>APPENDIX - TABLE 1</u> (continued)

2a. Geometric Probability - generalization of the random spheres model to approximately-spherical similarly-shaped convex particles. (Santaló (1955)) 2b. Mathematics - fractional differentiation and Weyl fractional transforms. (Erdélyi (1951), Sneddon (1966))	<u>Equation (T1.2)</u> $s(y) = k_1(y) \int_y^\infty \frac{k_2(x,y)u(x)}{(x-y)^\mu}\,dx,$ $0 \leq y \leq x \leq \infty,\ 0 \leq \mu < 1,$ where $k_1(y)$ and $k_2(x,y)$ are known non-singular functions on $[0,\infty)$.	($k_1(y) = \nu C N_V / 2\pi a_n N_A$, $k_2(x,y) = (x/a)^{\mu-1}$, where ν, C, N, N_A, a_n and μ are defined in Jakeman and Anderssen (1975b) and §2). Determine the size distribution of approximately-spherical similarly-shaped convex particles $g([x/a_n]^{\frac{1}{2}}) = u(x)$ in a three-dimensional opaque aggregate from observations of the size distribution of their areas of section $s(y)$ by random planes through the aggregate. ($k_1(y) = \Gamma(1+\mu)^{-1}$, $k_2(x,y) \equiv 1$) For suitably smooth functions $s(y)$, $-u(x)$ defines its μ-th fractional derivate. For suitably well behaved functions $u(x)$, $s(g)$ defines its Weyl fractional transform. A discussion of applications of these concepts to the solution of boundary value problems can be found in Sneddon (1966).
3a. Mathematics - fractional differentiation and Riemann-Liouville fractional transforms. (Atkinson(1974), Erdélyi (1951) and Sneddon(1966)) 3b. Various Applications - transformed version of equation (T1.4) below. (Weiss (1972))	<u>Equation (T1.3)</u> $s(y) = k_1(y) \int_0^y \frac{k_2(x,y)u(x)}{(y-x)^\mu}\,dx,$ $0 \leq x \leq y < \infty,\ 0 \leq \mu < 1,$ where $k_1(y)$ and $k_2(x,y)$ are known non-singular functions on $[0,\infty)$.	($k_1(y) = \Gamma(1+\mu)^{-1}$, $k_2(x,y) \equiv 1$) For suitably smooth functions $s(y)$, $u(x)$ defines its μ-th fractional derivative. For suitably well behaved functions $u(x)$, $s(y)$ defines its Riemann-Liouville fractional transform. A discussion of applications of these concepts to the solution of boundary value problems can be found in Sneddon (1966). ($\mu = \frac{1}{2}$) When appropriate, the equivalence of (T1.3) and (T1.4) can be used. When the x and y variables of (T1.3) are redefined to be $\bar{x}$ and $\bar{y}$, the equivalence is defined by $\bar{x} = a^2 - y^2$, $\bar{y} = a^2 - x^2$, with $\bar{y} \leq a$.

APPENDIX - TABLE 1 (continued)

4a. Physics - emission coefficients in radiation technology (Minerbo and Levy (1969) for a summary of references).	Equation (T1.4) $$s(y) = k_1(y) \int_y^a \frac{k_2(x,y)u(x)}{(x^2 - y^2)^{\frac{1}{2}}} dx,$$ $0 \leq y \leq x \leq a < \infty$, $k_1(x)$ known and non-singular.	($k_1(y) = 2$, $k_2(x,y) = x$) The determination in radiation technology of the emission coefficient $u(x)$ of an extended radiation source from externally measured radiance data $s(y)$.
4b. Mathematics - solution of first and second kind Cauchy integral equations. (Peters (1963,1968)		(The various forms of $k_1(y)$ and $k_2(x,y)$ are given in Peters (1963,1968)) The solution of first and second kind Cauchy integral equations is reduced to solving successively a pair of Abel-type integral equations. In the first equation, $s(y)$ is derived from the non-homogeneous term in the given first or second kind Cauchy equation while $u(x)$ corresponds to the solution of the second Abel-type equation.
4c. Various Applications - truncated version of equation (T1.1) above.		For many of the applications discussed for equation (T1.1), due to truncation (whether data generated or theoretical), its solution often reduces to the solution of (T1.4).
4d. Interferometry - analysis of the fringe shifts in interferograms. (Merzkirch(1974), Chapter VIII, §3.A)		($k_1(y) = 2/\lambda, \lambda$ = wavelength of light, $k_2(x,y) = x$) Determine the density of a radially symmetric flow $u(x)$ from externally measured fringe shifts $s(y)$ on interferograms of the flow obtained using an interferameter.
5a. Metallurgy - rod-like precipitates in metal alloys. (Santaló (1955))	Equation (T1.5) $$s(y) = \frac{4yV_FN_V}{S_FN_A} \int_y^\infty \frac{u(x)}{x(x^2-y^2)^{\frac{1}{2}}} dx,$$ $0 \leqslant y \leqslant x < \infty$, with V_F, N_V, S_F and N_A constants which depend on the type of rod-like particles being examined - see §2.	The determination of the size distribution of the rod-like precipitates $u(x)$ in a metal alloy from observations of the size distribution of the rod sections on random planes taken through the alloy (under the assumption that the alloy is transparent).

APPENDIX - TABLE 1 (continued)

6a. Biology - plate like cells (flat convex particles) in an opaque medium. (Santaló (1955))	**Equation (T1.6)** $$s(y) = \frac{\pi y N_V}{8N_A}\int_y^\infty \frac{u(x/2)}{(x^2-y^2)^{\frac{1}{2}}}\,dx,$$ $0 \leqslant y \leqslant x < \infty$, with a, N_V and N_A constants which depend on the flat circular particles (discs) being examined - see §2.	The determination of the size distribution of the circular discs $u(x/2)$ in some opaque biological medium from observations of the size distribution of their line intercepts $s(y)$ on random planes taken through the medium.
7a. Microscopy - the thin section model corresponding to the random spheres model. (Jakeman (1976), Goldsmith (1967)) 7b. Various Applications - replacement of random plane sections by random thin sections.	**Equation (T1.7)** $$s(y) + \lambda u(y) = k_1(y)\int_y^\infty \frac{k_2(x,y)u(x)}{(x^2-y^2)^{\frac{1}{2}}}\,dx,$$ $0 \leqslant y \leqslant x \leqslant \infty$, where $k_1(y)$ and $k_2(x,y)$ are known non-singular.	(λ = T (thickness of section), $k_1(y) = 2y/(2m+T)$, $m = \int_0^\infty xu(x)dx$, $k_2(x,y) \equiv 1$) Determine the size distribution of spheres $u(x)$ in a three dimensional aggregate from observations using transmission electron microscopy of the size distribution of circles $s(y)$ on random thin sections of the aggregate of thickness T. In the above examples (such as 1a, 1c, and 1d), the relevant equation now takes the form (T1.7) when random plane sections are replaced by random thin sections of thickness T, and the observed size distribution $s(y)$ is of circles or equivalent circles, so that $u(x)$ becomes a size distribution of spheres or equivalent spheres, respectively.

APPENDIX - TABLE 1 (continued)

8a. Brownian Motion - the calculation of the probability that a sample path of a Brownian motion process $w(t)\{0 \leq t \leq \infty\}$ crosses either the boundary $y=a(t)$ or $y = b(t)\,(b(0) < 0 < a(0)$ and $b(t) < a(t), 0 \leq t \leq T < \infty)$ before $t = T$ is a basic problem in many fields such as diffusion theory, gambler's ruin, collective risk, Kolmogorov-Smirnov statistics, cumulative-sum methods, sequential analysis and optimal stopping. (Durbin (1971), Anderssen, de Hoog, Weiss (1973))	**Equation (T1.8)** $P_a(t) = \int_0^t P_{aa}(t,s)d\alpha(s) + \int_0^t P_{ba}(t,s)d\beta(s) \quad (0 \leq t \leq T),$ $P_b(t) = \int_0^t P_{ab}(t,s)d\alpha(s) + \int_0^t P_{bb}(t,s)d\beta(s)$ where $P_c(t) = (2\pi t)^{-\frac{1}{2}}\exp[-\frac{1}{2}\{c(t)\}^2/t] \quad (c = a \text{ or } b),$ $P_{de}(t,s) = \{2\pi(t-s)\}^{-\frac{1}{2}} \exp[-\frac{1}{2}\{e(t) - d(s)\}^2/(t-s)] \quad (d,e = a \text{ or } b).$	(A system of Abel-type equations) Let $\alpha(t)$ = Probability {there exist $s_0 < s_1 < t$ for which the sample path $w(s)$ crosses $y = a(s)$ at s_0, and $w(s)$ crosses $y = b(s)$ for the first time at s_1}, and $\beta(t)$ = Probability {there exist $s_0 < s_1 < t$ for which $w(s)$ crosses $y = b(s)$ at s_0, $w(s)$ crosses $y = a(s)$ for the first time at s_1}. Then, if $a'(t)$ and $b'(t)$ exist and are bounded for $0 \leq t \leq T$, it follows from theory developed by Fortet (1943) (pp.217 and 220) and the properties of Brownian motion processes that the calculation of $\alpha(t)$ and $\beta(t)$ can be reduced to the solution of the given system of singular first kind Volterra equations.

APPENDIX - TABLE 1 (continued)

	Equation (T1.9)	
9a. Seismology - from observations in a spherically symmetric Earth of the epicentral arc of travel $\Delta(p)$ and the corresponding travel-time $T(p)$ for various earthquake waves with parameter p, calculate the elastic wave velocity $v(r)$ as a function of depth (radius, r). (Jeffreys (1962), Bateman (1910), Knott (1919), Macelwane (1951))	$s(p) = p \int_p^{R/V} \frac{\partial \log r/\partial u}{(u^2-p^2)^{\frac{1}{2}}} du,$ $p = dT/d\Delta$ = ray parameter, $s(p) = \Delta(p)/2,$ r = radius at which ray with parameter p bottoms, $u = r/v,$ $V = v(R),$ R = radius of Earth.	(The Bateman-Knott transformation of the non-linear travel-time equation in seismology) The integral equations which relate epicentral arc of travel $\Delta(p)$, travel time $T(p)$ and velocity $v(r)$ as a function of radius r are non-linear in $v(r)$, the unknown of interest. They are usually derived on the basis of Fermat's principle of least action which implies that the time of travel of any ray through the Earth must be stationary. Thus, equations for the path of travel, and hence $v(r)$, can be derived from the Euler-Lagrange equations for the functional which defines the time of travel. By redefining what the dependent and independent variables are (i.e., by introducing an appropriate change of variables), the basic non-linear equation becomes a linear Abel-type equation. (Jeffreys (1962), §2.05, Macelwane (1951), p. 273). The computational methods of solution applied to real data are based on this redefinition of variables.

APPENDIX - TABLE 2

Classification of Abel-type Integral Equations in Applications on the Basis of the Mathematical Rationale for Occurrence

Mathematical Rationale for Occurrence	Particular Form of the Abel-type Integral Equation	Related Subject Area and Context of Application
1. The probability $K(a,x)da$ that the area of intersection between a spherical particle of radius x and a random plane sectioning it, has a value between a and $a + da$ is (Santaló(1955)) $K(a,x)da=\{2x[\pi(a_n x^2-a)]^{\frac{1}{2}}\}^{-1}$ da, where a_n denotes the maximum taken over all possible values of a; and hence, on writing $a = \pi y^2$, $K(a,x)\ da = (x^2 - y^2)^{-\frac{1}{2}}dy$.	$s(y) = \frac{y}{m}\int_y^\infty u(x)/(x^2 - y^2)^{\frac{1}{2}}dx,$ $m = \int_0^\infty xu(x)\ dx,$ $0 \leqslant y \leqslant x \leqslant \infty.$	Applications 1a-d, 2a, 5a and 6a of Table 1; geometric probability and stereological applications in Physics, Chemistry, Metallurgy, Biology etc. (Underwood, de Wit and Moore (1976))
2. The definition of a fractional derivative and a fractional integral transform.	$s(y) = \Gamma(1+\alpha)^{-1}\int_y^\infty \frac{u(x)}{(x-y)^\alpha}\ dx,$ $s(y) = \Gamma(1+\alpha)^{-1}\int_0^y \frac{u(x)}{(y-x)^\alpha}\ dx,$ $0 \leqslant \alpha \leqslant 1.$	Applications 2b and 3a of Table 1.

APPENDIX - TABLE 2 (continued)

3. The equivalence of the kinetic energy and the loss of potential energy (of a falling particle); viz. (Lonseth (1977)) $\frac{1}{2} m (d\bar{s}/dt)^2 = gm(y - x)$, with m mass, g the Earth's gravitational constant, $\bar{s}$ arc length, t time, $y-x$ change in height.	$\int_0^y \frac{u(x)}{(y-x)^{\frac{1}{2}}} dx = (2g)^{\frac{1}{2}} s(y)$, $0 \leq x \leq y$, $u(x)$ is the curve the particle follows, and $s(y)$ is the time of travel as a function of height fallen.	The solution of the falling particle problem given by Abel in 1823; dynamics of particles.
4. Transformation of certain Fredholm integral equations which involve spherical, radial or circular symmetry; e.g. (Mazur and Wims (1966)) applying the transformation $t = (y^2-x^2)^{\frac{1}{2}}$, the first kind Fredholm equation $s(y) = 2\int_0^\infty W(t)u((x^2+t^2)^{\frac{1}{2}})dt$ is reduced to a first kind Abel-type equation.	$s(y) = 2\int_y^\infty \frac{xW[(x^2-y^2)^{\frac{1}{2}}]u(x)}{(x^2-y^2)} dx$, $0 \leq y \leq x \leq \infty$.	Application 1e of Table 1.

APPENDIX - TABLE 2 (continued)

5. Decomposition of complex problems into a sequence of simpler problems such as Abel-type equations; e.g. (Peters(1963)) replace the first kind Cauchy integral equation $\int_0^1 \frac{u(x)}{x-\xi}\,dx = f(\xi),\ 0 < \xi < 1,$ by the solution of a pair of Abel-type integral equations.	$\int_0^y \frac{\psi(t)}{(y-t)^{\frac{1}{2}}}\,dt = s(y) = \int_0^y \xi^{\frac{1}{2}} f(\xi)\,d\xi + 2cy^{\frac{1}{2}},$ $\int_t^1 \frac{x^{\frac{1}{2}}u(x)}{(x-t)^{\frac{1}{2}}}\,dx = \psi(t),$ $c = \int_0^1 u(x)\,dx,$ where $\psi(t)$ denotes the intermediate solution.	Application 4b of Table 1. (Peters (1963,1968))
6. Projection (Optically and X-ray) of radially symmetric (cylindrical and spherical) structure on a plane; e.g., if $u(r)$, $r^2 = x^2 + y^2$, denotes the absorption properties of a circular cylinder of radius a with centre at the origin, then the total absorption which can occur to a projected ray, which passes at a distance y from the centre of the cylinder, is $s(y) = \int_{-(a^2-y^2)^{\frac{1}{2}}}^{(a^2-y^2)^{\frac{1}{2}}} u(r)\,dx.$ Introducing the transformation $r^2 = x^2 + y^2$, this equation becomes a standard first kind Abel-type equation.	$s(y) = 2\int_y^a \frac{x\,u(x)}{(x^2-y^2)^{\frac{1}{2}}}\,dx,$ $0 \le y \le x \le a.$	Application 4a and 4d of Table 1; interferometry (Minerbo and Levy (1969), Merzkirch (1974), Chapter VIII, §3.A, p. 143); gravitational anomalies for an axially symmetric distribution of masses (Strahov and Maslennikova (1975)).

APPENDIX - TABLE 2 (continued)

7. A direct consequence of the nature of a Brownian motion process. Since at time t, the variance of a Brownian motion process (which starts at time s) is proportional to $t-s$, the Abel-type equations in (T1.8) correspond to the evaluation of normal integrals on such random processes.	Equations (T1.8) of Table 1.	Application 8a of Table 1.
8. Transformation of non-linear Volterra-type integral equations. When the kernel of a given integral equation contains a Volterra-type singularity of the form $[u^2(z) - p^2]^{-\frac{1}{2}}$ (i.e., for one of the limits of integration of z, $u^2(z)=p$) then a redefinition of u as the independent variable along with a corresponding redefinition of the dependent variable, will transform it to a (linear) first kind Abel-type equation.	$s(p) = p \int_p^{R/V} \frac{\partial \log r/\partial u}{(u - p)^{\frac{1}{2}}}\, du$, $p = dT/d\Delta$ = ray parameter, $s(p) = \Delta(p)/2$, r = radius at which ray with parameter p bottoms, $u = r/v$, $V = v(R)$, R = radius of Earth.	Application 9a of Table 1.

APPENDIX - TABLE 4

Table of K(a,x), Corresponding Form of (3.1), and Associated Size Distribution Estimation Problem

Data Category	Size Distribution Estimation Problem	$K(a,x)$	Corresponding Form of Equation (3.1)
(a)	From the size distribution, $s(y)$, of the radii of circular sections on random plane sections from an opaque field of spheres, determine the size distribution, $u(x)$, of the radii of the spheres.	$K(a/x^2, 1)x^2$ $= \{2x\ [\pi(a_n x^2 - a)]^{\frac{1}{2}}\}^{-1}$	$s(y) = y/m \int_y^\infty \frac{u(x)}{(x^2-y^2)^{\frac{1}{2}}} dx,\ m = \int_0^\infty xu(x)dx.$
(a)	From the size distribution, $s(y)$, of the semi-major (or minor) axes of elliptical sections on random plane sections through an opaque field of prolate or oblate ellipsoids, determine the size distribution, $u(x)$, of the semi-major (or minor) axes of the ellipsoids.	$K(a/x^2, 1)/x^2$ $= \{2x\ [\pi(a_n x^2 - a)]^{\frac{1}{2}}\}^{-1}$	$s(y) = y/m \int_y^\infty \frac{u(x)}{(x^2-y^2)^{\frac{1}{2}}} dx,\ m = \int_0^\infty xu(x)dx.$
(a)	From the size distribution, $s(a)$, of the parameter (e.g. area) which specifies the planar convex figures on random plane sections through an opaque field of approximately-spherical similarly-shaped one-parameter convex particles, determine the size distribution, $u(x)$, of the convex particles in terms of the size distribution of their parameter.	$\nu \left\{x^2[(a_n x^2 - a)/x^2]^{\mu}\right\}^{-1}$, $\int_0^{a_n} K(a, 1)\, da = 1$, $\int_0^{a_n} aK(a, 1)\, da = 2\pi\ V/C.$	$s(a) = \frac{\nu C N_V}{4\pi a_n N_A} \int_a^\infty \left(\frac{x}{a_n}\right)^{\mu-1} \frac{g(\sqrt{x / a_n})}{(x-a)^{\mu}} dx.$

APPENDIX - TABLE 4 (continued)

(b)	From the size distribution, $s(y)$, of the length of line intercepts on random plane sections through an opaque field of flat (no thickness) circular discs, determine the size distribution, $u(x)$, of the diameters of the circular discs.	$K(a/x,1)/x$, with $K(a,1) = \begin{cases} \frac{1}{2}a(4-a^2)^{-\frac{1}{2}}, & a < 2, \\ 0, & a \geq 2. \end{cases}$	$s(y) = \frac{\pi y N_V}{8N_A} \int_y^\infty \frac{u(x/2)}{(x^2-y^2)^{\frac{1}{2}}}\, dx.$
(c)	From the size distribution, $s(y)$, of the length of line intercepts on random lines probing through an opaque field of spheres, determine the size distribution, $u(x)$, of the diameters of the spheres.	$K(a/x,1)/x$ with $K(a,1) = a/2.$	$s(2y) = -\frac{\pi N_V y}{N_L} \int_y^\infty u(x)\, dx.$
(c)	From the size distribution, $s(y)$, of the length of line intercepts on random lines probing through an opaque field of approximately spherical, similarly-shaped one-parameter convex particles, determine the size distribution, $u(x)$, of the convex particles in terms of their parameter.	$(\nu a/x)^\mu/x$, $\int_0^{a_n} K(a,1)\, da = 1,$ $\int_0^{a_n} aK(a,1)\, da = 4V/S.$	$s(a_n y) = -\frac{(\nu a_n)^\mu S N_V y}{4N_L} \int_y^\infty x^{1-\mu}\, u(x)\, dx.$

APPENDIX - The Algorithm

An algorithm has been implemented and copies of the FORTRAN program are available from either R. S. Anderssen (Computer Centre) or A. J. Jakeman (CRES), Australian National University, P. O. Box 4, Canberra ACT2600, Australia. At the moment, it is only implemented for the random spheres model (see Appendix - Table 1, application 1b). A general package for Abel-type integral equations along with test data is in preparation. It will include algorithms for the direct evaluation of linear functionals defined on the data.

REFERENCES

R.S. Anderssen (1973) Computing with noisy data with an application to Abel's equation, in *Error, Approximation and Accuracy* (eds. F.R. de Hoog and C.L. Jarvis), pp. 61-68, University of Queensland Press, Brisbane, Queensland, 1973.

R.S. Anderssen (1976) Stable procedures for the inversion of Abel's equation, *JIMA* 17 (1976), 329-342.

R.S. Anderssen (1977) Some numerical aspects of improperly posed problems or why regularization works and when not to use it, *Tech. Report No.* 52, *Australian National University, Computer Centre*, January, 1977.

R.S. Anderssen and P. Bloomfield (1974a) A time series approach to numerical differentiation, *Technometrics* 16 (1974), 69-75.

R.S. Anderssen and P. Bloomfield (1974b) Numerical differentiation procedures for non-exact data, *Numer. Math.* 22 (1974), 157-182.

R.S. Anderssen, F.R. de Hoog and R. Weiss (1973) On the numerical solution of Brownian motion processes, *J. Appl. Prob.* 10 (1973), 409-418.

R.S. Anderssen and A.J. Jakeman (1975a) Product integration for functionals of particle size distributions, *Utilitas Mathematica* 8 (1975), 111-126.

R.S. Anderssen and A.J. Jakeman (1975b) Abel type integral equations in stereology. II. Computational methods of solution and the random spheres approximation, *J. Micros.* 105 (1975), 135-153.

K.E. Atkinson (1974) An existence theory for Abel integral equations, *SIAM J. Math. Anal.* 5 (1974), 729-736.

K.E. Atkinson (1975) *An Introduction to Numerical Analysis*, Preprint, Mathematics, University of Iowa, Iowa City, Iowa, 1975.

A.V. Baev and V.B. Glasko (1976) The solution of the inverse kinematic problem of seismology by means of a regularizing algorithm (Russian) *Zh. Vychisl. Mat. i Mat. Fiz.* 16 (1976), 922-931.

H. Bateman (1910) The solution of the integral equation which connects the velocity of propagation of an earthquake wave in the interior of the Earth with the times which the disturbance takes to travel to different stations of the Earth's surface, *Phil. Mag.* 19 (1910), 576-587.

P. Baudhuim and J. Berthet (1967) Electron microscopic examination of subcellular fractions, *J. Cell. Biol.* 35 (1967), 631-648.

P. Bloomfield (1976) *Fourier Analysis of Time Series: An Introduction*, Wiley Series in Probability and Mathematical Statistics, Wiley, 1976.

H. Brunner (1973) The numerical solution of a class of Abel integral equations by piecewise polynomials, *J. Comp. Phys.* 12 (1973), 412-416.

H. Brunner (1974) Global solution of the generalized Abel integral equation by implicit interpolation, *Math. Comp.* 28 (1974), 61-67.

J. Cullum (1971) Numerical differentiation and regularization, *SIAM J. Numer. Anal.* 8 (1971), 254-265.

P. Davy (1977) Projected thick sections through multidimensional particle aggregates, to appear.

P. Davy and R.E. Miles (1977) Sampling theory for opaque spatial specimens, to appear.

F.R. de Hoog and R. Weiss (1973) Asymptotic expansions for product integration, *Math. Comp.* 27 (1973), 295-306.

J. Durbin (1971) Boundary crossing probabilities for the Brownian motion and Poisson processes and techniques for computing the power of the Kolmogorov-Smirnov test, *J. Appl. Prob.* 8 (1971), 431-453.

D. Elliott (1977) *Lagrange Interpolation - Decline and Fall?* An Invited Address given at the 21st Annual Conference of the Australian Mathematical Society, May 19, 1977, La Trobe University; *Technical Report* No 87, Mathematics, University of Tasmania, May, 1977.

A. Erdelyi (1954) *Bateman Manuscript Project: Tables of Integral Transforms,* Volume II, McGraw-Hill, New York, 1954.

R. Fortet (1943) Les functions aléatoires du type de Markov associées a certaines équations lineaires aux derivées partielles du type parabolique, *J. Math. Pures Appl.* 22(1943), 177-243.

P.L. Goldsmith (1967) The calculation of true particle size distributions from the sizes observed in a thin slice, *Brit. J. Appl. Phys.* 18 (1967), 813-830.

E.J. Hannan (1970) *Multiple Time Series,* John Wiley, New York, 1970.

J.E. Hilliard (1968) Direct determination of the moments of the size distribution of particles in an opaque field, *Trans. Metal. Soc. AIME* 242 (1968), 1373-1380.

J.E. Hilliard and L. Rickels (1977) Unfolding of the size distribution of Mn S inclusions in free machining steels, *Private Communication,* March, 1977.

P.A.W. Holyhead and S. McKee (1976) Stability and convergence of multistep methods for linear Volterra integral equations of the first kind, *SIAM J. Numer. Anal.* 13 (1976), 269-292.

P.A.W. Holyhead and S. McKee (1977) Stability and convergence of multistep methods for the generalized Abel's integral equation (Part I), submitted

R.T. de Hoff and R.N. Rhines (Editors) (1968), *Quantitative Microscopy,* McGraw-Hill, New York, 1968.

E.D. Hyan and J. Nutting (1956) The tempering of plain carbon steels, *J. Iron and Steel Inst.* 148 (1956), 148-165.

A.J. Jakeman (1975) *Numerical Inversion of Abel Type Integral Equations in Stereology*, Ph. D. Thesis, Australian National University, August, 1975.

A.J. Jakeman (1976) *Numerical Inversion of a Second Kind Singular Volterra Equation - The Thin Section Equation of Stereology*, unpublished manuscript.

A.J. Jakeman and R.S. Anderssen (1975) Abel type integral equations in stereology. I. General discussion, *J. Micros.* 105 (1975), 121-133.

A.J. Jakeman and R.L. Scheaffer (1977) On the properties of product integration estimators for linear functionals of particle size distributions, submitted.

G.M. Jenkins and D.G. Watts (1968) *Spectral Analysis and Its Application*, Holden-Day, San Francisco, 1968.

H. Jeffreys (1962) *The Earth* (Fourth Edition) Cambridge University Press, Cambridge, 1962.

N. Keiding, S.T. Jensen and L. Ranek (1972) Maximum likelihood estimation of the size distribution of liver cell nuclei from the observed distribution in a plane section, *Biometrics* 28 (1972), 813-829.

C.C. Knott (1919) The propagation of earthquake waves through the Earth and connected problems, *Roy. Soc. Edinburgh Proc.* 39 (1919), 157-208.

L.H. Koopmans (1974) *The Spectral Analysis of Time Series*, Academic Press, New York, 1974.

P. Linz (1967) Application of Abel transforms to the numerical solution of problems in electrostatics and elasticity, *MRC Technical Summary Report* #826, University of Wisconsin, Madison, Wis., 1967.

P. Linz (1968) *Numerical Methods for Volterra Integral Equations*, Ph.D. Thesis, University of Wisconsin, Madison, Wis., 1968.

A.T. Lonseth (1977) Sources and applications of integral equations, *SIAM Review* 19 (1977), 241-278.

J.B. Macelwane (1951) Evidence on the interior of the Earth derived from seismic sources, P. 227-304 in B. Gutenberg (Ed.) *Internal Constitution of the Earth*, Dover, New York, 1951.

J. Mazur (1971) Numerical solution of integral equation of the first kind applied to slit correction in small angle x-ray scattering, *J. Research Nat. Bureau Standards* 75B (1971), 173-187.

J. Mazur and A.M. Wims (1966) A formal solution for slit correction in small-angle x-ray scattering, *J. Research Nat. Bureau Standards* 70A (1966), 467-471.

J. Meisner (1967) Estimation of the distribution of diameters of spherical particles from a given grouped distribution of diameters of observed circles formed by a plane section, *Statistica Neerlandica* 21 (1967), 11-30.

W. Merzkirch (1974) *Flow Visualization*, Academic Press, New York and London, 1974.

R.E. Miles and P. Davy (1977a) On the choice of quadrats in stereology, to appear.

R.E. Miles and P. Davy (1977b) Precise and genera[illegible]nditions for the validity of a comprehensive set of stereological fundamental formulae, to appear.

G.N. Minerbo and M.E. Levy (1969) Inversion of Abel's integral equation by means of orthogonal polynomials, *SIAM J. Numer. Anal.* 6 (1969), 598-616.

P.A.P. Moran (1972) The probabilistic basis of stereology, *Special Supplement to Adv. Appl. Prob.* 4 (1972), 69-91.

O.H. Nestor and H.N. Olsen (1960) Numerical methods for reducing line and surface probe data, *SIAM Review* 2 (1960), 200-207.

W.L. Nicholson (Editor) (1972) Proceedings of the Symposium on Statistical and Probabilistic Problems in Metallurgy, Seattle, Washington, 4-6 August, 1971, in *Special Supplement to Adv. Appl. Prob.* 4 (1972).

B. Noble, *Lecture Notes*, 1970.

A.S. Peters (1963) A note on the integral equation of the first kind with a Cauchy kernel, *Comm. Pure Appl. Math.* 16 (1963), 51-61.

A.S. Peters (1968) Abel's equation and the Cauchy integral equation of the second kind, *Comm. Pure Appl. Math.* 21 (1968), 51-65.

R. Piessens and P. Verbaeten (1973) Numerical solution of the Abel integral equation, *BIT* 13 (1973), 451-457.

S.A. Saltikov (1967) The determination of the size distribution of particles in an opaque material from a measurement of the size distribution of their section, in *Stereology*, ed. H. Elias, Springer, New York, p.163-173, 1967.

L.A. Santaló (1953) *Introduction to Integral Geometry*, Hermann, Paris (Act. Sci. Indust. No. 1198), 1953.

L.A. Santaló (1955) Sobre la distribución de los tamanos de corpúsculos contenidos en un cuerpo a partir de la distribución en sus secciones o proyecciones, *Trabajos Estadist* 6 (1955), 181-196.

I. Sneddon (1966) *Mixed Boundary Value Problems in Potential Theory*, North-Holland Publishers, Amsterdam, 1966.

V.N. Strahov and A.I. Maslennikova (1975) The interpretation of gravitational anomalies of an axially symmetric distribution of masses (Russian), *Perm. Gos. Univ. Ucen. Zap.* # 357 (1975), 52-62.

G.M. Tallis (1970) Estimating the distribution of spherical and elliptical bodies in conglomerates from plane sections, *Biometrics* 26 (1970), 87-103.

E.E. Underwood (1970) *Quantitative Stereology,* Addison-Wesley, Mass., 1970.

E.E. Underwood, R. de Wit and G.A. Moore (Editors) (1976) Fourth International Congress for Stereology, *National Bureau of Standards Special Publication* #431, U.S. Government Printing Office, Washington, Jan, 1976.

G. Wahba (1976) A survey of some smoothing problems and the method of generalized cross-validation for solving them, *Tech. Report No* 457 *Statistics, University of Wisconsin-Madison,* July, 1976.

G. Wahba (1977) Practical approximate solutions to linear operator equations when the data are noisy, *SIAM J. Num. Anal.*

G.S.Watson (1971) Estimating functionals of particle size distributions, *Biometrika* 58 (1971), 483-490.

E.R. Weibel, W. Staübli, H.R. Gnägi and F.A. Hess (1969) Correlated morphometric and biochemical studies on the liver cell, *J. Cell. Biol.* 42 (1969), 68-91.

R. Weiss (1972) Product integration for the generalized Abel equation, *Math. Comp.* 26 (1972), 177-190.

R. Weiss and R.S. Anderssen (1972) A product integration method for a class of singular first kind Volterra equations, *Numer. Math.* 18 (1972), 442-456.

P. Whittle (1952) Some results in time series analysis, *Skand. Aktuar* 35 (1952), 48-60.

S.D. Wicksell (1925) The corpuscle problem, *Biometrika* 17 (1925), 84-99.

S.D. Wicksell (1926) The corpuscle problem II, *Biometrika* 18 (1925), 151-172.

A. Young (1954) The application of approximate product-integration to the numerical solution of integral equations, *Proc. Roy. Soc. London Ser.* A 224(1954), 561-573.

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A commonly occurring class of integral equations which arise regularly in applications are the separable first kind Abel-type integral equations such as

$$(*) \qquad s(y) = \int_y^\infty k_1(y)k_2(x)\,(x^2 - y^2)^{-\mu}u(x)dx, \quad 0 < \mu < 1, \quad x \geq y \geq 0 .$$

Though the mathematical properties of this equation (such as conditions for the

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existence, uniqueness and smoothness of its solutions, its improperly posed nature, the existence of inversion formulas, etc.) have been examined in some detail in the literature, its numerical solution poses a number of difficulties especially when $s(y)$ is only available as discrete observational data

$$\{d_i\} = \{d_i = s(x_i) + \varepsilon_i,\ i = 1,2,\ldots,n;\ \varepsilon_i \text{ discrete random errors}\}\ .$$

In this paper~~, we (i)~~ reviews the numerous applications in which the solution of the Abel-type integral equation ~~equations like (*)~~ for discrete observational data $\{d_i\}$ is the basic step, compares ~~(ii)~~ with respect to given discrete observational data $\{d_i\}$, ~~compare~~ the use of pseudo-analytic methods and the direct evaluation of its inversion formulas as a basis for solving ~~(*)~~ this equation, ~~(iii)~~ proposes a specific algorithm based on ~~the~~ these conclusions~~, of (ii)~~, and ~~(iv)~~ examines the consequences of the fact that, for ~~(*)~~ the equation, linear functionals defined on its solution $u(x)$ can be redefined as linear functionals on the data $s(y)$. The justification for the latter is that, in applications involving separable first kind Abel-type integral equations, inferences are usually based on (linear) functionals defined on $u(x)$, not on $u(x)$ itself. This point is illustrated with an example from metallurgy.